Business PreCalculus MATH 1643 Section 004, Spring 2014

Lesson 11: The Coordinate Plane

Definition 1. **The Coordinate Plane:** *A pair of real numbers in which the order is specified is called an* ordered pair *of real numbers. The ordered pair* (x, y) *has* first component x *and* second component y. *The sets of ordered pairs of real numbers are identified with points on a plane called the* coordinate plane *or the* cartesian plane.

Definition 2. **Distance Formula in the Coordinate Plane:** *Let* $P = (x_1, y_1)$ *and* $Q = (x_2, y_2)$ *be any two points in the coordinate plane. Then the distance between* P *and* Q*, denoted by* $d(P, Q)$*, is given by the* distance formula

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 1. *Find the distance between the points* $P = (-2, 5)$ *and* $Q = (3, -4)$.
Solution:

$$\begin{aligned} d(P, Q) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ d(P, Q) &= \sqrt{(3 - (-2))^2 + (-4 - 5)^2} \\ &= \sqrt{(3 + 2)^2 + (-9)^2} \\ &= \sqrt{25 + 81} \\ &= \sqrt{106} \end{aligned}$$

Definition 3. **Midpoint Formula:** *The coordinates of the midpoint* $M = (x, y)$ *of the line segment joining* $P = (x_1, y_1)$ *and* $Q = (x_2, y_2)$ *are given by*

$$M = (x, y) = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}).$$

Example 2. *Find the midpoint of the line segment joining* $P = (-3, 6)$ *and* $Q = (1, 4)$.
Solution:

$$\begin{aligned} M &= (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}) \\ M &= (\frac{(-3) + 1}{2}, \frac{6 + 4}{2}) \\ &= (\frac{-2}{2}, \frac{10}{2}) \\ &= (-1, 5) \end{aligned}$$

Definition 4. **Finding the Intercepts of the Graph of an Equation:**

Step 1. *To find the* x-intercepts *of the graph of an equation,* set $y = 0$ *in the equation and* solve for x.

Step 2. *To find the* y-intercepts *of the graph of an equation,* set $x = 0$ *in the equation and* solve for y.

The concept of ***symmetry*** helps us to sketch graphs of equations. A graph has symmetry with respect to a line L, if one portion of the graph is a ***mirror image*** of another portion in L.

Definition 5. Tests for Symmetries:

1. *A graph is* symmetric with respect to *the* y-axis, *if for every point* (x, y) *on the graph, the point* $(-x, y)$ *is also on the graph.*

2. *A graph is* symmetric with respect to *the* x-axis, *if for every point* (x, y) *on the graph, the point* $(x, -y)$ *is also on the graph.*

3. *A graph is* symmetric with respect to *the* origin, *if for every point* (x, y) *on the graph, the point* $(-x, -y)$ *is also on the graph.*

Example 3. *Determine whether the graph of* $x^3 + y^2 - xy^2 = 0$ *is symmetric with respect to the* y*-axis, or* x*-axis, or both.*
Solution: *First we will check* ***symmetry with respect to the*** y***-axis by replacing*** x ***with*** $-x$.

$$\begin{aligned} x^3 + y^2 - xy^2 &= 0 \quad \textit{original equation} \\ (-x)^3 + y^2 - (-x)y^2 &= 0 \quad \textit{replacing } x \textit{ with } -x \\ -x^3 + y^2 + xy^2 &= 0 \quad \textit{simplifying} \end{aligned}$$

Since $-x^3 + y^2 + xy^2 = 0$ *is a different equation than the original one when* x *is replaced with* $-x$*, then the graph is not symmetric with respect to the* y*-axis. Next, we examine* ***symmetry with respect to the*** $x-$***axis by replacing*** y ***with*** $-y$.

$$\begin{aligned} x^3 + y^2 - xy^2 &= 0 \quad \textit{original equation} \\ x^3 + (-y)^2 - x(-y)^2 &= 0 \quad \textit{replacing } y \textit{ with } -y \\ x^3 + y^2 - xy^2 &= 0 \quad \textit{simplifying: } (-y)^2 = y^2 \end{aligned}$$

Hence, we obtain the original equation when y *is replaced with* $-y$*. Thus the graph of* $x^3 + y^2 - xy^2 = 0$ *is symmetric with respect to the* x*-axis.*

Definition 6. Circle: *A* circle *is a set of points in a Cartesian coordinate plane that are at a fixed distance* r *from a specified point* (h, k)*. The fixed distance* r *is called the* radius *of the circle, and the specified point* (h, k) *is called the* center *of the circle. The equation of a circle with center* (h, k) *and radius* r *is*

$$(x - h)^2 + (y - k)^2 = r^2,$$

and is called the **standard form** *of an equation of a circle.*

Definition 7. General Form of the Equation of a Circle: *The* **general form** *of the equation of a circle is*

$$x^2 + y^2 + ax + by + c = 0.$$

The graph of this equation is a circle, provided $d > 0$*, where* $d = \frac{1}{4}(a^2 + b^2 - 4c)$.

Definition 8. **Converting the General Form of a Circle into Standard Form:**

Step 1. *Group the x-terms and y-terms separately.*

Step 2. *Complete the square on both the x- and the y-terms by adding $(\frac{a}{2})^2$ and $(\frac{b}{2})^2$ to both sides of the equation.*

Step 3. *Factor and Simplify.*

Step 4. *Rewrite in standard form.*

Example 4. *Find the center and the radius of the circle with equation*

$$x^2 + y^2 - 6x + 8y + 10 = 0.$$

Solution:

$$\begin{aligned}
x^2 + y^2 - 6x + 8y + 10 &= 0 && \textit{original equation}\\
x^2 - 6x + y^2 + 8y + 10 &= 0 && \textit{grouping the } x\textit{- and the } y\textit{-terms}\\
x^2 - 6x + (\frac{-6}{2})^2 + y^2 + 8y + (\frac{8}{2})^2 + 10 &= (\frac{-6}{2})^2 + (\frac{8}{2})^2 && \textit{adding } (\frac{a}{2})^2 \textit{ and } (\frac{b}{2})^2 \textit{ to both sides}\\
x^2 - 6x + 9 + y^2 + 8y + 16 + 10 &= 9 + 16\\
(x^2 - 6x + 9) + (y^2 + 8y + 16) + 10 - 10 &= 25 - 10\\
(x-3)^2 + (y+4)^2 &= 15 && \textit{factoring}\\
(x-3)^2 + (y-(-4))^2 &= (\sqrt{15})^2 && \textit{Rewrite in standard form}
\end{aligned}$$

Hence, the center of $x^2 + y^2 - 6x + 8y + 10 = 0$ is $(3, -4)$ and its radius is $\sqrt{15}$.